Fenton's sum of translates approach for classical minimax questions of approximation theory

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The classical problem of Chebyshev asks for placement of n roots in the interval $[0, 1]$, such that the maximum of the arising monic algebraic polynomial, obtained by multiplying together the root factors, has least possible maximum norm. Four decades ago Bojanov formulated a variant of the question, assuming that the n roots are not simple root factors, but multiple roots of given multiplicity (and thus the product is not a degree n, but degree N algebraic polynomial). Studying these questions from a new point of view, we first take the logarithm of the absolute value of these products, and thus obtain a sum of logarithms. Choosing the roots, or nodes, means translating of the arising logarithmic terms. Having a zero translates into a singularity attaining minus infinity.

With this changed setup we loose the well-known vector space property of our polynomial space. The sums of the arbitrarily translated "log root factors", however, has a different, remarkable structure. We will call these log root factors "kernels", and generally even assume that they may be quite different. We only assume that the kernels are concave (as is the case with $log|t|$), and perhaps that they are singular (attaining minus infinity at zero). A further advantage is that we can relatively easily incorporate an "external field function" J, which after exponentiating back becomes a multiplicative weight function $w(x) := \exp(J(x))$. Thus we can handle even weighted minimax questions of the sort.

We analyse what conditions are needed to regain the usual basic results of approximation theory. Between the endpoints 0 and 1 and the roots or nodes y_j, there arise $n + 1$ intervals, and maxima on these intervals will be denoted by m_j. The question is when the maximum of the m_j will be minimal? It turns out that under quite general assumptions the equioscillating property (that all m_j are equal) is necessary and sufficient, and it uniquely characterizes the extremal configuration. Moreover, we will find that any prescription of the differences between the m_j is also uniquely attained, and so the n-dimensional difference vector of these interval maxima is in a continuous homeomorphism correspondence - in fact, in a bi-Lipschitz fashion - with the totality of admissible or regular n-dimensional vectors of nodes. This is remarkable in particular in cases when J can be singular on vast sets, and admissible nodes form a complicated subset of R^n.

We will give examples to demonstrate that our general conditions are all indeed indispensable, and derive new observations even in the most classical case

of the Chebyshev Problem. On our way we will explore estimates for one-sided derivatives of the m_j even in cases of non-differentiable kernels and field functions, find best regularity properties of the m_j as functions of the node systems, apply a generalization of Clarke to the inverse mapping theorem holding for Lipschitz (but not necessarily differentiable) functions, and will draw a number of approximation theoretic consequences like the weighted version of Bojanov's Problem or the trigonometric case.