From Padua points to "fake" nodes: old, new and open problems

Constructive Approximation of Functions 2 - 22-24 Sept. 2021

Stefano De Marchi
University of Padova (Italy)

August 27, 2021

Abstract

Padua points, miracolously discovered in 2005 in Padova [1, 2, 3], are the first set of points on the square $[-1,1]^2$ that are explictely known, unisolvent for total degree polynomial interpolation and with Lebegsue constant increasing like $\log^2(n)$ of the degree. Lagrange interpolation at the Padua points has been recently used in several scientific and technological applications, for example in Computational Chemistry (the Fun2D subroutine of the CP2K simulation package for Molecular Dynamics, see paper), in Image Processing (algorithms for image retrieval by colour indexing), in Materials Science (Modelling of Composite Layered Materials, see paper), in Mathematical Statistics (Copula Density Estimation, see abstract by L. Qu, p. 67), in Quantum Physics (Quantum State Tomography, see paper); moreover, it has been added in the *Chebfun2 package.*

One of the key features of the *Padua Points* is that they lie on a particular *Lissajous curve.* The more general topic of multivariate Polynomial Approximation on Lissajous Curves turned out to be of interest in the emerging field of Magnetic Particle Imaging (MPI) (see, e.g., some recent publications and the activities of the scientific network MathMPI). Lissajous sampling seems to be relevant also in the field of Atomic Force Microscopy (AFM). Padua points have been recently used for solving PDEs with RBF.

Other important properties of Padua points are

1. In 2d, Padua points are a WAM for interpolation and for extracting Approximate Fekete Points and Discrete Leja sequences.
2. In 3d, Padua points can be used for constructing tensor product WAMs on different compacts.

Unfortunately their extension to higher dimensions still the biggest open problem.

The concept of mapped bases has been widely studied (cf. e.g. [4] and references therein), which turns out to be equivalent to map the interpolating nodes and then construct the approximant in the classical form without the need of resampling. The mapping technique is general, in the sense that works with any basis and can be applied to continuous, piecewise or discontinuous functions or even images. All the proposed methods show convergence to the interpolant provided that the function is resampled at the mapped nodes. In applications, this is often physically unfeasible. An effective method for interpolating via mapped bases in the multivariate setting, referred as *Fake Nodes Approach* (FNA), has been presented in [5]. I will present some interesting connection of the AFN with Padua points and "families of relatives

nodes", that can be used as "fake nodes" for multivariate approximation, discussing some open problems.

There are many people to thanks for this work. In particular, **thanks to** L. Bos, M. Caliari, A. Sommariva, S. Waldron, M. Vianello, Y. Xu and many others with who I have worked in the last two decades. The Polish friends are among them. With their collaboration we proved these interesting results.

References

[1] Caliari, Marco; Bos, Len; De Marchi, Stefano; Vianello, Marco; Xu, Yuan (2006) "Bivariate Lagrange interpolation at the Padua points: the generating curve approach", J. Approx. Theory, 143 (1): 15–25

[2] CAA Padova-Verona Research Group on Constructive Approximation webpage: `https://www.math.unipd.it/~marcov/CAApadua.html`

[3] Wikipedia: Padua points `https://en.wikipedia.org/wiki/Padua_points`

[4] S. De Marchi, F. Marchetti, E. Perracchione, D. Poggiali, "Polynomial interpolation via mapped bases without resampling", J. Comput. Appl. Math. 364 (2020), 112347.

[5] S. De Marchi, F. Marchetti, E. Perracchione, D. Poggiali, "Multivariate approximation at fake nodes", Appl. Math. Comput. 391 (2021), 125628.